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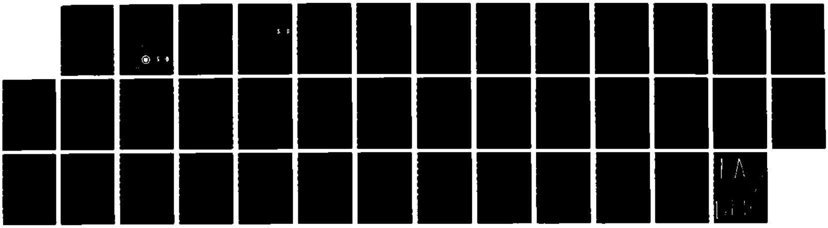

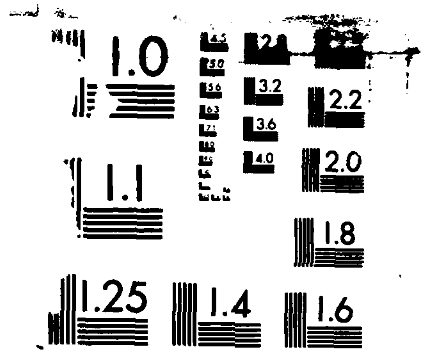

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ROBUST OPTIMUM INVARIANT TESTS OF COVARIANCE STRUCTURES USEFUL IN LINEAR MODELS

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Baltimore County

August 1986

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Key Words and Phrases: Covariance structures, null robustness, nonnull robustness, optimality robustness, LBI test, BLUE, LRT.

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ROBUST OPTIMUM INVARIANT TESTS OF COVARIANCE STRUCTURES USEFUL IN LINEAR MODELS

1. Introduction.

In this paper we investigate robust optimum invariant tests of some covariance structures that naturally arise in the context of robustness study in linear models. The concept of robustness in connection with linear models is entirely different from the notion prevalent in multivariate analysis (vide Kariya and Sinha (1985)) and refers to the structures of X and V in the model $(\underset{\sim}{Y}, X\underset{\sim}{\beta}, \sigma^2 V)$ in contrast to the distribution of $\underset{\sim}{Y}$. Here X is known as the design matrix and $\sigma^2 V$ the variance-covariance matrix of $\underset{\sim}{Y}$.

To describe this concept, let $(\underset{\sim}{Y}, X\underset{\sim}{\beta}, \sigma^2 I)$ be the assumed (probably incorrect) model while $(\underset{\sim}{Y}, X\underset{\sim}{\beta}, \sigma^2 V)$ be the correct model, resulting in the specification error in the dispersion matrix. Then it is well known that the BLUEs of all estimable linear parametric function $A\underset{\sim}{\beta}$ remain the same under both the models if and only if the following condition holds on the structure of V:

$$X'VZ = 0 \tag{1.1}$$

where Z denotes a matrix of maximal rank satisfying the condition $Z'X = 0$. This result, in various equivalent forms, appears in Rao (1967), Zyskind (1967), Rao and Mitra (1971), Mathew and Bhimasankaram (1983), and also in Sinha and Drygas (1983). Our object is to test the null hypothesis that V possesses the structure ~~satisfying (1.1)~~ $X'VZ=0$ based on samples on $\underset{\sim}{Y}$ under the model $(\underset{\sim}{Y}, X\beta, \sigma^2 V)$ for a fixed design matrix X. This hypothesis is of considerable interest as its acceptance greatly simplifies determination of BLUEs of estimable linear parametric functions.

beta

Below we work with a canonical form of this problem which is now developed. Let $\underset{\sim}{Y}$: n x 1, X: n x k with rank (X) $= r \leq k$, so that $X = X^{o}C$ for some X^{o}: n x r of rank r and for some C: r x k of rank r. Consequently the matrix Z: n x (n−r) which satisfies $Z'X = 0$ also satisfies $Z'X^{o} = 0$. It is then clear that the condition (1.1) is equivalent to

$$X^{o'}VZ = 0 \tag{1.2}$$

Defining $\underset{\sim}{Y}_{(1)} = Z'\underset{\sim}{Y}$ and $\underset{\sim}{Y}_{(2)} = X_0'\underset{\sim}{Y}$ and making the 1:1 transformation $\underset{\sim}{Y} \rightarrow (\underset{\sim}{Y}_{(1)}, \underset{\sim}{Y}_{(2)})'$ it then follows that the condition (1.2) is equivalent to testing the hypothesis that $\underset{\sim}{Y}_{(1)}$ and $\underset{\sim}{Y}_{(2)}$ are uncorrelated. If $\underset{\sim}{Y}$ is assumed to be normally distributed, this is the familiar problem of test of independence of two random vectors $\underset{\sim}{Y}_{(1)}$ and $\underset{\sim}{Y}_{(2)}$ with the added restriction that $E\underset{\sim}{Y}_{(1)} = 0$ since $Z'X^{o} = 0$. This problem is analyzed via invariance in the next section where normality of the underlying data matrix is replaced by an elliptically symmetric distribution.

There is another form of robustness in linear models in connection with tests of estimable linear parametric functions. To describe this briefly, consider the problem of testing H_0: $A\beta = 0$ under the assumed model, $(Y, X\beta, \sigma^2 I)$, σ^2 unknown, and Y is distributed normally. Here $A\beta$ is estimable and hence testable. It is well-known that the F-test based on the ratio of sums of squares due to the hypothesis and due to the error is both LPT and UMPI under a suitable group of transformations (vide Lehmann (1959)). The answers to the question "Is the F-test (given above) under the model $(Y, X\beta, \sigma^2 I)$ still LPT under the correct model $(Y, X\beta, \sigma^2 V)$?" have been put forward by Khatri (1980), Ghosh and Sinha (1980) and Mathew and Bhimasankaram (1983). It turns out that the answer is in the affirmative under the following condition on V;

$$(I - P_{X_0})\, V (I - P_{X_0}) = a(I - P_{X_0}) \qquad (1.3)$$

for some $a > 0$ where $P_A = A(A'A)^- A'$, $(A'A)^-$ is a generalized inverse of $A'A$, and $X_0 = X(I - A^- A)$.

The second object in this paper is indeed to test the hypothesis that V in the model $(Y, X\beta, \sigma^2 V)$ possesses a structure satisfying (1.3), for a fixed linear parametric function $A'\beta$, based on samples of Y for a fixed design matrix X. Incidentally, if we demand (1.3) to hold for all estimable $A'\beta$, it turns out that $V = I$ is the only matrix satisfying this condition.

As before, here again we work with a canonical form. Writing $I - P_{X_o} = DD'$ for some D: $n \times \gamma$ of rank $\gamma = \text{rank}\,(I - P_{X_o})$ and noting that $D'D$ is p.d., it follows easily that (1.3) is equivalent to

$$D'VD = aI, \qquad \text{for some } a > 0 \tag{1.4}$$

Defining now $W = D'Y$, it follows that (1.4) is equivalent to testing the sphericity of $\underset{\sim}{W}$ with the added restriction that $E(\underset{\sim}{W}_{(1)}) = 0$ where $\underset{\sim}{W} = (\underset{\sim}{W}'_{(1)}\underset{\sim}{W}'_{(2)})'$. This follows from the fact that the range or the column space of D contains a subspace which is orthogonal to X. This problem is taken up in section 3 via invariance with normality of $\underset{\sim}{Y}$ replaced by elliptical symmetry of $\underset{\sim}{Y}$.

2. Test of Independence

The canonical form of this problem is as follows. Based on a data matrix Y: $n \times p\ [\underset{n \times p_1}{X} : \underset{n \times p_2}{Z}]$ obeying the model

$$Y = [\underset{\sim}{1}\,\underset{\sim}{\mu}' : 0] + U\textstyle\sum^{1/2}, \quad \underset{\sim}{\mu} \in R^{p_1}, \quad \textstyle\sum \text{ p.d.} \tag{2.1}$$

where U has an elliptically symmetric distribution with density

$$f(u) = q(\mathrm{tr}\, u'u) \qquad \text{for some } q\text{: } [0, \infty) \to [0, \infty) \tag{2.2}$$

such that $\int_{R^{n\times p}} q(\mathrm{tr}\, u'u)\,du = 1,$

we want to test the hypothesis H_0: $\Sigma_{12} = 0$ vs. H_1: $\Sigma_{12} \neq 0$. Here Σ is expressed as

$$\Sigma = \begin{array}{cc} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} & \begin{array}{c} p_1 \\ p_2 \end{array} \\ \begin{array}{cc} p_1 & \quad p_2 \end{array} & \end{array}$$

. The p.d.f of Y can be written as

$$f(Y|\mu,\Sigma) = |\Sigma|^{-n/2} \cdot q(\mathrm{tr}\Sigma^{-1}(X - \underset{\sim}{1}\mu',Z)'(X - \underset{\sim}{1}\mu',Z)) \tag{2.3}$$

When $\underset{\sim}{\mu} = \underset{\sim}{0}$ or the mean of Z is not known and Y is normal, this is the usual problem of testing independence of two vectors for which optimum solutions do exist in the literature. For example, Schwartz (1967) established that the test based on tr $S_{xz}S_{zz}^{-1}S_{zx}S_{xx}^{-1}$ is IBI in general. For nonnormal Y, its null and optimality robustness under certain conditions on q are established in Kariya and Sinha (1985). Of course, when $p_1 = p_2 = 1$, this test boils down to the ordinary produt moment correlation test and becomes UMPI. Here S denotes the sample Wishart matrix based on Y and S is

decomposed as
$$S = \begin{array}{cc} \begin{bmatrix} S_{xx} & S_{xz} \\ S_{zx} & S_{zz} \end{bmatrix} & \begin{array}{c} p_1 \\ p_2 \end{array} \\ \begin{array}{cc} p_1 & p_2 \end{array} & \end{array} .$$

We shall see how the solution changes in our problem because of the information that the mean of Z is 0. We mention that under the assumption of normality of Y, this problem under a slightly wider framework appears in Eaton and Kariya (1983).

Before we discuss this problem from the point of view of invariance, let us quickly look into the LRT. Define $\bar{x}: p_1 \times 1$, $\bar{z}: p_2 \times 1$ and S in the usual fashion and decompose S as $S = \begin{bmatrix} S_{xx} & S_{xz} \\ S_{zx} & S_{zz} \end{bmatrix}$ as mentioned before. The likehood function (2.3) can be written as

$$f(\mu,\Sigma|Y) = |\Sigma|^{-n/2} q(n\, tr\Sigma^{-1} \begin{pmatrix} \bar{x}-\mu \\ \bar{z} \end{pmatrix} ((\bar{x}-\mu)', \bar{z}') + tr\Sigma^{-1}S) \qquad (2.4)$$

Assuming that $q(\cdot)$ is a nonincreasing function of its argument, it follows that the MLE $\hat{\mu}$ of μ satisfies $\bar{x} - \hat{\mu} - \Sigma_{11}^{-1} \Sigma_{12}\bar{z} = 0$. This yields

$$\sup_{\underset{\sim}{\mu}} f(\mu, \underset{\sim}{\Sigma} | Y) = |\Sigma|^{-n/2} q(n \text{ tr } \Sigma_{22}^{-1} \underset{\sim\sim}{\bar{z}\bar{z}'} + \text{tr } \Sigma^{-1} S) \tag{2.5}$$

Using (Rao(1973))

$$\begin{vmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{vmatrix}^{-1} = \left|\begin{array}{c|c} \Sigma_{11.2}^{-1} & -\Sigma_{11.2}^{-1}\Sigma_{12}\Sigma_{22}^{-1} \\ \hline -\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11.2} & \Sigma_{22}^{-1}+\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11.2}^{-1}\Sigma_{12}\Sigma_{22}^{-1} \end{array}\right| \tag{2.6}$$

where

$$\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \quad ,$$

and

$$|\Sigma| = |\Sigma_{22}| \, |\Sigma_{11.2}| \quad ,$$

we get

$$\text{tr } \Sigma^{-1} S + n \text{ tr } \Sigma_{22}^{-1} \underset{\sim\sim}{\bar{z}\bar{z}'} = \text{tr } \Sigma_{22}^{-1} (S_{11} + n \underset{\sim\sim}{\bar{z}\bar{z}'}) + \tag{2.7}$$

$$\text{tr } \Sigma_{11.2}^{-1}(S_{xx} - \Sigma_{12}\Sigma_{22}^{-1}S_{zx} - S_{xz}\Sigma_{22}^{-1}\Sigma_{21} + \Sigma_{12}\Sigma_{22}^{-1}S_{zz}\Sigma_{22}^{-1}\Sigma_{21}) \; .$$

While maximizing the likelihood function with respect to Σ, we consider the reparametrization Σ_{22}, $\Sigma_{11.2}$ and $\Sigma_{22}^{-1}\Sigma_{21} = \xi$ (say). The expression in (2.5) in terms of these new parameters can be written as

$$\sup_{\underset{\sim}{\mu}} f(\underset{\sim}{\mu}, \Sigma | y) = |\Sigma_{22}|^{-n/2} \, |\Sigma_{11.2}|^{-n/2} \quad . \tag{2.8}$$

$$q(\mathrm{tr}\Sigma_{22}^{-1}(S_{zz}+n\underset{\sim}{\bar{z}}\underset{\sim}{\bar{z}}')+\mathrm{tr}\Sigma_{11.2}^{-1}(s_{xx\cdot z}+(\xi-S_{xz}S_{zz}^{-1})S_{zz}(\xi-S_{xz}S_{zz}^{-1})'))$$

Clearly this attains its maximum with respect to ξ when $\xi = S_{xz}S_{zz}^{-1}$ resulting in

$$\sup_{\underset{\sim}{\mu},\xi} f(\underset{\sim}{\mu}, \Sigma_{22}, \Sigma_{11.2}, \xi | y) = |\Sigma_{22}|^{-n/2} \, |\Sigma_{11.2}|^{-n/2} \quad . \tag{2.9}$$

$$q(\mathrm{tr}\ \Sigma_{22}^{-1}\ (S_{zz} + n\underset{\sim}{\bar{z}}\underset{\sim}{\bar{z}}') + \mathrm{tr}\ \Sigma_{11.2}^{-1}S_{xx\cdot z})$$

Finally, using a result of Anderson and Fang(1982) we know that if q is nonincreasing and differentiable the MLEs of Σ_{22} and $\Sigma_{11.2}$ are given by

$$\hat{\Sigma}_{22} = \lambda_{max}(q) \cdot (S_{zz} + n\underset{\sim}{\bar{z}}\underset{\sim}{\bar{z}}'),$$

$$\hat{\hat{\Sigma}}_{11.2} = \lambda_{max}(q)\ S_{xx\cdot z}$$

where $\lambda_{max}(q)$ is the solution of the equation

$$q'(\frac{p}{\lambda}) + \frac{n\lambda}{2}\, q(\frac{p}{\lambda}) = 0 .$$

For example, if $q(x) = e^{\frac{-x}{2}}$, $\lambda_{max}(q) = \frac{1}{n}$.

Therefore, we have,

$$\sup_{\underset{\sim}{\mu},\Sigma} f(\underset{\sim}{\mu},\xi,\Sigma_{22},\Sigma_{11.2}|y) = \lambda_{max}^{-n}(q)\,|S_{zz}+n\underset{\sim}{\bar{z}}\underset{\sim}{\bar{z}}'|^{-n/2}\,|S_{xx\cdot z}|^{-n/2} \cdot$$

$$q(p_1\lambda_{max}^{-1}(q) + p_2\lambda_{max}^{-1}(q)) \qquad (2.10)$$

$$= \lambda_{max}^{-n}(q)\ |S_{zz} + n\underset{\sim}{\bar{z}}\underset{\sim}{\bar{z}}'|^{-n/2}\ |S_{xx\cdot z}|^{-n/2}\ q(p\lambda_{max}^{-1}(q))$$

Analogously, under the null hypothesis H_0: $\Sigma_{12} = 0$, we get,

$$\sup_{\substack{\mu,\Sigma \\ H_0}} f(\underset{\sim}{\mu},\Sigma|y)=\lambda_{max}^{-n}(q)\,|S_{zz}+n\bar{z}\bar{z}'|^{\frac{-n}{2}}\,|S_{xx}|^{\frac{-n}{2}}q(p\lambda_{max}^{-1}(q)) \qquad (2.11)$$

yielding the LRT statistic as $\{|S_{xx\cdot z}|/|S_{xx}|\}^{n/2}$. In the above we have assumed that q satisfies $q(p\lambda_{max}^{-1}(q)) < \infty$, $0 < \lambda_{max}(q) < \infty$.

<u>Remark</u> 2.1: It may be noted that the LRT derived above is just the one without the information that Z has mean zero and so it ignores this information. When Y is normally distributed, this is derived in Lee and Geisser (1972). As noted in Eaton and Kariya (1983), this is rather surprising.

Our derivation of the LBI test of H_0: $\Sigma_{12} = 0$ versus H_1: $\Sigma_{12} \neq 0$ under the model (2.3) parallels a similar derivation in Kariya and Sinha (1985) where Z, like X, also is assumed to have an unknown mean matrix of the form $\underset{\sim}{1}\underset{\sim}{\delta}'$ for some $\underset{\sim}{\delta} \in R^{p_2}$. We restrict our attention to the likelihood function given in (2.4) and without any loss of generality due to the invariance of the problem assume that a) $\bar{\underset{\sim}{x}} - \underset{\sim}{\mu} = \underset{\sim}{0}$ in (2.4) and b) Σ is of the form

$$\Sigma(\delta) = \begin{bmatrix} I_{p_1} & \Gamma \\ \Gamma' & I_{p_2} \end{bmatrix} \text{ where}$$

$\Gamma = (\Delta, 0)$: $p_1 \times p_2$ with $\Delta = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_{p_1})$. Here without any loss of generality $p_1 \leq p_2$ is assumed. The result a) follows upon noting that the testing problem remains invariant under the transformation $\bar{\underset{\sim}{x}} \to \bar{\underset{\sim}{x}} + \underset{\sim}{\xi}$ for $\underset{\sim}{\xi} \in R^{p_1}$, that the left invariant measure on R^{p_1} is Lebesgue, and that the result of integrating out $\underset{\sim}{\xi}$ in (2.4) after the substitution $\bar{\underset{\sim}{x}} \to \bar{\underset{\sim}{x}} + \underset{\sim}{\xi}$ (this is while invoking Wijsman's

Representation Theorem) is nothing but to put $\bar{x} - \mu = 0$ in (2.4), multiply the right hand side of (2.4) by $|\Sigma_{11.2}|^{1/2}$, apart from a constant, and replace q by some $\tilde{q}$: $[0, \infty) \to [0, \infty)$ satisfying a similar integrability condition as q. The resultant expression for (2.4) is then given by (writing q for $\tilde{q}$)

$$f(Y|\Sigma)=k|\Sigma_{22}|^{-n/2}|\Sigma_{11.2}|^{-(n-1)/2}q(n\ \mathrm{tr}\Sigma_{22}^{-1}\bar{z}\bar{z}'+\mathrm{tr}\Sigma^{-1}S) \qquad (2.12)$$

with $\Sigma = \begin{bmatrix} I_{p_1} & \Gamma \\ \Gamma' & I_{p_2} \end{bmatrix}$, where k is a constant.

In this setup the problem is to test H_0: $\Delta = 0$ versus H_1: $\Delta \neq 0$. It is easy to see that the testing prolem in this somewhat reduced form remains invariant under the group G of transformations

$G = A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$, $A_i \in G\ell(p_i)$, with the group action

$$S \to ASA' \quad , \quad \bar{z} \to A_2\bar{z} \qquad (2.13)$$

A left invariant measure on G is $\nu(dg) = \nu_1(dA_1)\nu_2(dA_2)$ with $\nu_i(dA_i) = |A_iA_i'|^{-(p_i+1)/2}$, $i = 1, 2$, and the inverse of the jacobian of transformation is given by $|A_1A_1'|^{\frac{n}{2}} \cdot |A_2A_2'|^{\frac{n+1}{2}}$.

Using Wijsman's Representation Theorem (1967), the ratio $r_\Delta(y) \equiv \frac{dP_\Delta}{dP_0}$ is obtained as

$$r_\Delta(y) =$$

$$\frac{\int_{G\ell(p_1)\times G\ell(p_2)} |\Sigma|^{-\frac{n}{2}} q(ntr\bar{z}'A_2'A_2\bar{z} + tr\Sigma^{-1}ASA')|A_1A_1'|^{\frac{n-p_1-1}{2}} |A_2A_2'|^{\frac{n-p_2}{2}} dA_1dA_2}{\int_{G\ell(p_1)\times G\ell(p_2)} q(ntr\bar{z}'A_2'A_2\bar{z} + tr\Sigma^{-1}ASA')|A_1A_1'|^{\frac{n-p_1-1}{2}} |A_2A_2'|^{\frac{n-p_2}{2}} dA_1dA_2} \tag{2.14}$$

<u>Remark 2.2</u>: It is clear that an optimum test of H_0 versus H_1 is obtained by examining the behavior of the ratio $r_\Delta(y)$ as a function of Δ. In the special case when $\bar{z}$ is absent in this ratio, it is proved in Kariya and Sinha (1985) that there exists a UMPI test if $p_1 = p_2 = 1$ and an LBI test in general. However, with the presense of $\bar{z}$ in this ratio, there does not exist a UMPI test even when $p_1 = p_2 = 1$. This is

observed in Eaton and Kariya (1983) when Y is distributed normally.

Now to derive an LBI test, we proceed as follows: Straightforward computations yield

$$\Sigma^{-1} = \begin{bmatrix} I_{p_1} & \Gamma \\ \Gamma' & I_{p_2} \end{bmatrix}^{-1} = \begin{bmatrix} I_{p_1} + \Gamma(I_{p_2} - \Gamma'\Gamma)^{-1}\Gamma' & -\Gamma(I_{p_2} - \Gamma'\Gamma)^{-1} \\ -(I_{p_2} - \Gamma'\Gamma)^{-1}\Gamma' & (I_{p_2} - \Gamma'\Gamma)^{-1} \end{bmatrix} \qquad (2.15)$$

$$= \left| \begin{matrix} I_{p_1} + \Delta(I_{p_1} - \Delta'\Delta)^{-1}\Delta' & -(\Delta(I_{p_1} - \Delta'\Delta)^{-1} : \quad 0) \\ \begin{bmatrix} -(I_{p_1} - \Delta\Delta')^{-1}\Delta' \\ 0 \end{bmatrix} & \begin{bmatrix} (I_{p_1} - \Delta'\Delta)^{-1} & 0 \\ 0 & I_{p_2 - p_1} \end{bmatrix} \end{matrix} \right|$$

and with $\Gamma = (\Delta : 0)$, and $||\Delta||^2 = \sum_{i=1}^{p_1} \delta_i^2$ small,

$$(I_{p_1} - \Delta\Delta')^{-1} = (I_{p_1} - \Delta'\Delta)^{-1} = I_{p_1} + \Delta\Delta' + o(||\Delta||^2) \qquad (2.16)$$

$$(I_{p_1} - \Delta\Delta')^{-1}\Delta = \Delta + o(||\Delta||^2) = \Delta(I_{p_1} - \Delta'\Delta)^{-1}$$

$$I_{p_1} + \Delta(I_{p_1} - \Delta'\Delta)^{-1}\Delta' = I_{p_1} + \Delta\Delta' + o(||\Delta||)^2$$

Also we have,

$$\text{tr } ASA' = \text{tr } A_1S_{11}A_1' + \text{tr } A_2S_{22}A_2' \tag{2.17}$$

so that

$$\text{tr}n\, \bar{z}'A_2'A_2\bar{z} + \text{tr } ASA' = \text{tr } A_1S_{11}A_1' + \text{tr } A_2(S_{22}+n\bar{z}\bar{z}')A_2' \tag{2.18}$$

Finally, using (2.15) and (2.16), we get

$$n \text{ tr } \bar{z}'A_2'A_2\,\bar{z} + \text{tr } \Sigma^{-1} ASA' \tag{2.19}$$

$$= [\text{tr } A_1S_{11}A_1' + \text{tr } A_2(S_{22}+n\bar{z}\bar{z}')A_2'] + \text{tr}(\Sigma^{-1}-I_p)ASA'$$

while

$$\text{tr}(\Sigma^{-1}-I_p)ASA' = \text{tr}(\Sigma^{-1}-I_p) \begin{bmatrix} A_1S_{11}A_1' & A_1S_{12}A_2' \\ A_2S_{21}A_1' & A_2S_{22}A_2' \end{bmatrix} \tag{2.20}$$

$$= \operatorname{tr}\ \left\{\left(\begin{bmatrix} \Delta\Delta' & -\Gamma \\ -\Gamma & \Gamma'\Gamma \end{bmatrix} + o^*(||\Delta||^2)\right) \begin{bmatrix} A_1S_{11}A_1' & A_1S_{12}A_2' \\ A_2S_{21}A_1' & A_2S_{22}A_2' \end{bmatrix}\right\}$$

$$= \operatorname{tr}\ \Delta\Delta' A_1S_{11}A_1' - 2\operatorname{tr}\ \Gamma A_2\ S_{21}A_1' + \operatorname{tr}\ \Gamma'\Gamma\ A_2S_{22}A_2' +$$

$$o\ (||\Delta||^2)\ (\operatorname{tr}\ ASA')$$

where $o^*(||\Delta||^2)$ is a matrix of order p x p all of whose elements are $o(||\Delta||^2)$. We now make the transformation

$$A_1S_{11}^{1/2} \to A_1 \quad , \quad A_2(S_{22} + n\ \underline{\underline{\bar{z}\bar{z}'}})^{1/2} \to A_2 \tag{2.22}$$

This reduces $r_\Delta(y)$ to

$r_\Delta(y) =$

$$\frac{|\Sigma|^{-\frac{n}{2}} \int_{G\ell(p_1)\times G\ell(p_2)} q(\operatorname{tr}\ A_1A_1' + \operatorname{tr} A_2A_2') + \operatorname{tr}\Delta\Delta' A_1A_1' - 2\operatorname{tr}\Gamma A_2WA_1' + \operatorname{tr}\Gamma'\Gamma A_2VA_2' + o(||\Delta||^2))\,|A_1A_1'|^{\frac{n-p_1-1}{2}}\,|A_2A_2'|^{\frac{n-p_2}{2}}\,dA_1dA_2}{\int_{G\ell(p_1)\times G\ell(p_2)} q(\operatorname{tr}\ A_1A_1' + \operatorname{tr}A_2A_2')\,|A_1A_1'|^{\frac{n-p_1-1}{2}}\ \,|A_2A_2'|^{\frac{n-p_2}{2}}\ dA_1dA_2}$$

where

$$V = (S_{22} + n\, \underset{\sim}{\bar{z}}\underset{\sim}{\bar{z}}')^{-1/2}\ S_{22}(S_{22} + n\, \underset{\sim}{\bar{z}}\underset{\sim}{\bar{z}}')^{-1/2}$$

$$W = (S_{22} + n\, \underset{\sim}{\bar{z}}\underset{\sim}{\bar{z}}')^{-1/2}\ S_{21}S_{11}^{-1/2} \quad ,$$

and the term $o(||\Delta||^2)$ is uniformly so in Y. This is because both V and W satisfy $||W'W|| < 1$, $||V|| < 1$.

We now expand the numerator of $r_\Delta(y)$ around $\Delta = 0$ using standard Taylor expansion. Towards this end, we assume that q is thrice continuously differentiable and

$$\int |q^{(i)}(\mathrm{tr}\ A_1A_1' + \mathrm{tr}\ A_2A_2')| \, |A_1A_1'|^{\frac{n-1-p_1}{2}} |A_2A_2'|^{\frac{n-p_2}{2}} |\mathrm{tr}\ PA_2QA_1'|^{2\ell+1}$$
$$|\mathrm{tr}\ PP'A_1A_1' + \mathrm{tr}\ RR'A_2A_2'|^3\, dA_1 dA_2 < \infty \ , \tag{2.23}$$

for $\ell = 0,1$, P: $p_1 \times p_2$, Q: $p_2 \times p_1$, R: $p_2 \times p_1$

where $q^i(x) = \dfrac{d^i q(x)}{dx^i}$, $\quad i = 1, 2, 3.$

Then,

$$q[(\mathrm{tr}\ A_1A_1' + \mathrm{tr}\ A_2A_2') + \mathrm{tr}\ \Delta\Delta' A_1A_1' - 2\mathrm{tr}\ \Gamma\ A_2WA_1' + \tag{2.24}$$

$$\operatorname{tr} \Gamma'\Gamma\, A_2VA_2' + o(||\Delta||^2)]$$

$$= q(\operatorname{tr} A_1A_1' + \operatorname{tr} A_2A_2') + \delta(A:\Delta)q^{(1)}(\operatorname{tr} A_1A_1' + \operatorname{tr} A_2A_2')$$

$$+ \frac{1}{2}\{\delta(A:\Delta)\}^2 q^{(2)}(\operatorname{tr} A_1A_1' + \operatorname{tr} A_2A_2') + \frac{1}{6}\{\delta(A:\Delta)\}^3 \cdot$$

$$q^{(3)}[(\operatorname{tr} A_1A_1' + \operatorname{tr} A_2A_2') + (1-\alpha)\,\delta(A:\Delta)] \quad , \quad 0 < \alpha < 1$$

where

$$\delta(A:\Delta) = \operatorname{tr} \Delta\Delta' A_1A_1' - 2\operatorname{tr} \Gamma\, A_2WA_1' + \operatorname{tr} \Gamma'\Gamma\, A_2VA_2' + o(||\Delta||^2)$$

To evaluate the integrals of these terms over $G\ell(p_1) \times G\ell(p_2)$, we note the fact that the integrals of odd functions of A_1 and A_2 are zero because the integrals are finite by our assumption (2.23). Moreover,

$$\int_{G\ell(p_1)\times G\ell(p_2)} (\operatorname{tr} \Gamma'\Gamma\, A_2VA_2')q^{(1)}(\operatorname{tr} A_1A_1' + \operatorname{tr} A_2A_2'), \qquad (2.25)$$

$$|A_1A_1'|^{\frac{n-1-p_1}{2}} |A_2A_2'|^{\frac{n-p_2}{2}} dA_1dA_2 = c_1(q)(\operatorname{tr}\Gamma'\Gamma)(\operatorname{tr}V) = c_1(q)(\operatorname{tr}\Delta'\Delta)(\operatorname{tr}V)$$

where

$$c_1(q)=\frac{1}{p_1p_2}\int_{G\ell(p_1)\times G\ell(p_2)}(\mathrm{tr}[I_{p_1}:O]A_2A_2'\begin{bmatrix}I_{p_1}\\O\end{bmatrix})q^{(1)}(\mathrm{tr}A_1A_1'+\mathrm{tr}A_2A_2')\,.\tag{2.26}$$

$$|A_1A_1'|^{\frac{n-1-p_1}{2}}\cdot|A_2A_2'|^{\frac{n-p_2}{2}}\,dA_1dA_2$$

and

$$\int_{G\ell(p_1)\times G\ell(p_2)}(\mathrm{tr}\ \Gamma\ A_2WA_1')^2q^{(2)}(\mathrm{tr}\ A_1A_1'+\mathrm{tr}\ A_2A_2')\,.\tag{2.27}$$

$$|A_1A_1'|^{\frac{n-1-p_1}{2}}|A_2A_2'|^{\frac{n-p_2}{2}}dA_1dA_2=c_2(q)(\mathrm{tr}\Gamma'\Gamma)(\mathrm{tr}WW')=c_2(q)(\mathrm{tr}\Delta'\Delta)(\mathrm{tr}WW')$$

where

$$c_2(q)=\frac{1}{p_1^{\;2}}\int_{G\ell(p_1)\times G\ell(p_2)}\left(\operatorname{tr}\,(I_{p_1}:O)A_2\begin{bmatrix}I_{p_1}\\ O\end{bmatrix}A_1'\right)^2 q^{(2)}(\operatorname{tr}A_1A_1'+\operatorname{tr}A_2A_2')$$

$$|A_1A_1'|^{\frac{n-1-p_1}{2}}\cdot|A_2A_2'|^{\frac{n-p_2}{2}}\,dA_1dA_2 \tag{2.28}$$

(2.25) and (2.27) can be proved along the same lines as in Kariya (1978), Eaton and Kariya (1983), and Kariya and Sinha (1985). The expressions for c_1 and c_2 and Y is normally distributed appear in Eaton and Kariya are given by $c_1^* = -\frac{n}{p_2}$, $c_2^* = \frac{n(n-1)}{p_1p_2}$.

Additionally, we get

$$\int_{G\ell(p_1)\times G\ell(p_2)}(\operatorname{tr}\,\Delta\Delta' A_1A_1'+\operatorname{tr}\Gamma'\Gamma A_2VA_2')^2 q^{(2)}(\operatorname{tr}A_1A_1'+\operatorname{tr}A_2A_2')\cdot \tag{2.29}$$

$$|A_1A_1'|^{\frac{n-1-p_1}{2}}|A_2A_2'|^{\frac{n-p_2}{2}}\,dA_1dA_2 \equiv o(||\Delta||^2)\ ;$$

$$\int_{G\ell(p_1)\times G\ell(p_2)}(\operatorname{tr}\,\Delta\Delta' A_1A_1'+\operatorname{tr}\Gamma'\Gamma A_2VA_2')(\operatorname{tr}\Gamma A_2WA_1')^2 q^{(2)}(\operatorname{tr}A_1A_1'+\operatorname{tr}A_2A_2')\cdot$$

$$|A_1A_1'|^{\frac{n-1-p_1}{2}}|A_2A_2'|^{\frac{n-p_2}{2}}\,dA_1dA_2 = o(||\Delta||^2)\ ; \tag{2.30}$$

and so on. These results follow primarily because V and W are bounded in norm as mentined before and the integrals involved are finie by our assumption (2.23).

We are now ready to collect all the different terms arising out of the integrals of the expression in (2.24). A straightforward computation shows that the ratio $r_\Delta(y)$ in (2.22) is given by

$$r_\Delta(y)=|\Sigma|^{-n/2}[1+(\mathrm{tr}\Delta\Delta')\{\mathrm{tr}V)c_1+c_2(\mathrm{tr}WW')\}+o(||\Delta||^2)] \qquad (2.31)$$

$$= 1+(\mathrm{tr}\ \Delta\Delta')\{c_1(\mathrm{tr}\ V) + c_2(\mathrm{tr}\ WW') + \frac{n}{2}\} + o(||\Delta||^2)$$

since

$$|\Sigma|^{-n/2} = |I_{p_1} - \Delta\Delta'|^{-n/2} = 1 + \frac{n}{2}\ \mathrm{tr}\ \Delta\Delta' + o(||\Delta||^2) \quad .$$

A simple application of the Neyman-Pearson Lemma then yields the following result.

<u>Theorem</u> 2.1: For testing H_0: $\Sigma_{12} = 0$ vs H_1: $\Sigma_{12} \neq 0$ under the model (2.1)-(2.2), the test which rejects H_0 for large values of $c_1(q)\mathrm{tr}\ V+ c_2(q)\ \mathrm{tr}\ WW'$ is LBI for a given q satisfying Assumption (2.23).

<u>Remark 2.3</u>: When V is absent, the LBI test statistic coincides with the popular expression tr $S_{22}^{-1}S_{21}S_{11}^{-1}S_{12}$ given by Kariya and Sinha (1985), and represents a robust LBI test for all q satisfying Assumption (2.23). When V is present but Y is normal, this expression is the same as in Easton and Kariya (1983) (their equation (4.6)).

<u>Remark 2.4</u>: The LBI test statistic derived for a specific q remains robust for $q \in Q$, a class of densities satisfying Assumption (2.23), whenever $\frac{c_1(q)}{c_2(q)}$ is a constant, independent of q. It is easy to verify that for normal variance mixtures

$$f(u) = \int_0^\infty \frac{e^{-\frac{1}{2}\mathrm{tr}\, u'u/w}}{(2\pi)^{np/2} w^{np/2}}\, dG(w),$$

$c_1(G)$ and $c_2(G)$ are given by $c_1(G) = c_1^*$, $c_2(G) = c_2^*$, independent of G. Hence the LBI test is optimality robust at least for arbitrary normal variance mixture family. The null robustness of the LBI test in this case follows easily from Kariya (1981).

3. <u>Testing Sphericity</u>

The canonical form of this problem is identical with that in Section 2. However, here we are testing $H_0: \Sigma = \sigma^2 I_p$ versus $H_1: \Sigma \neq \sigma^2 I_p$, $\sigma^2 > 0$ unknown. When the mean of X is also zero or the mean of Z is unknown, this is the well known problem of testing sphericity for which optimum tests are derived in Sugiura (1972) under the assumption of normality of Y, and in Kariya and Sinha (1985) under a more general distribution of Y. Here, as in

Section 2, only one of the means is unknown and we shall see the solution changes drastically.

Before we employ the principle of invariance in an attempt to derive an optimum invariant test, here also we first derive the LRT. The likelihood function appears in (2.4) and its unconstrained supremum is given in (2.10) of Section 2. Under the null hypothesis H_0: $\Sigma = \sigma^2 I_p$, $\sigma^2 > 0$ unknown, (2.5) reduces to

$$\sup_{\mu} f(\mu, \sigma^2|y) = (\sigma^2)^{-np/2} q(tr(n\, \bar{z}\bar{z}' + s)/\sigma^2) \tag{3.1}$$

Finally, using a version of the same result by Anderson and Fang(1982) mentioned in Section 2, we know that if q is nonincreasing and differentiable the MLE of σ^2 is given by

$$\hat{\sigma}^2 = \Theta(q)\ tr(n\, zz' + s)$$

where $\Theta(q)$ is the solution of the equation

$$q'(\tfrac{1}{\theta}) + \frac{np\theta}{2} q(\tfrac{1}{\theta}) = 0$$

For example, if $q(x) = e^{\frac{-x}{2}}$, $\theta(q) = \frac{1}{np}$.

Hence.

$$\sup_{\substack{\mu,\sigma^2 \\ H_0}} f(\mu,\sigma^2|Y)=\{\theta(q)\}^{-np/2}\{tr(n\,\bar{z}\bar{z}'+s)\}^{-np/2} q\left(\frac{1}{\theta(q)}\right) \tag{3.2}$$

Comparing (2.10) and (3.2), the LRT criterion λ is obtained as

$$\lambda \;\alpha\; \left[\frac{|S_{xx\cdot z}|\;|S_{zz}+n\,\bar{z}\bar{z}'|}{\{tr(S_{xx}+S_{zz}+n\,\bar{z}\bar{z}')\}^{p}}\right]^{\frac{n}{2}} \tag{3.3}$$

Here also we have assumed that $\lambda_{max}^{-n}(q)q(p\lambda_{max}^{-1}(q)) < \infty$ and $\theta(q)^{\frac{-np}{2}}\, q\left(\frac{1}{\theta(q)}\right) < \infty$

Thus the LRT criterion remains robust as long as q is nonincreasing and differentiable.

To derive an optimum invariant test, we note that the testing problem H_0: $\sum = \sigma^2 I$ versus H_1: $\sum \neq \sigma^2 I$, $\sigma^2 > 0$, unknown under the model (2.1)-(2.2) remains invariant under the group G of transformations $G = R_+ \times R^{p_1} \times O(p_1) \times O(p_2)$ acting on Y as

$$g(Y) = g[X: Z] = c[(XH_1 + \underset{\sim}{1}\,\underset{\sim}{\delta}') : ZH_2] \tag{3.4}$$

for $g = (c, \underset{\sim}{\delta}, H_1, H_2) \varepsilon G$. A left invariant measure $\nu(dg)$ on G is given by $\frac{dc}{c} d\underset{\sim}{\delta}\nu(dH_1)\nu(dH_2)$ where $d\underset{\sim}{\delta}$ is Lebesgue on R^{p_1} and $\nu(dH_i)$ is the invariant probability measure on $O(p_i)$, $i = 1, 2$. A straightforward calculation shows that the ratio $dP^T_{H_1}/dP^T_{H_0}$ of nonnull to null distribution of a maximal invariant T is given by

$$\frac{dP^T_{H_1}}{dP^T_{H_0}} = |\Sigma|^{-\frac{n-1}{2}} |\Sigma_{22}|^{-1/2} \int_{O(p_1)\times O(p_2)} (1+F)^{-\frac{np}{2}} \nu(dH_1)\nu(dH_2) \qquad (3.5)$$

where

$$F = \frac{tr(\Sigma^{-1} - I_{p_1})HSH' + tr(\Sigma^{-1}_{22} - I_{p_2})H_2VH_2'}{tr\ S + tr\ V} \qquad (3.6)$$

and

$$V = n\,\underset{\sim}{\bar{z}}\underset{\sim}{\bar{z}}', \quad H = \begin{bmatrix} H_1 & O \\ O & H_2 \end{bmatrix} : p \times p \qquad (3.7)$$

We note that when V is absent in (3.5), the ratio boils down to the familiar expression (Kariya and Sinha (1985))

$$\frac{dP^T_{H_1}}{dP^T_{H_0}} = |\Sigma|^{-\frac{n}{2}} \int_{\theta(p)} (1 + F^*)^{-np/2} \, \nu(dH) \tag{3.8}$$

where

$$F^* = \frac{\mathrm{tr}(\Sigma^{-1} - I_p)HSH'}{\mathrm{tr}\ S} \quad \text{and} \quad H \in O(p)$$

However, in our problem, because of presence of $\bar{z}$ and the structure of the joint density in (2.4), H has to be taken as a block orthogonal matrix given in (3.7) above.

Remark 3.1: It is interesting to observe that the ratio $\frac{dP^T_{H_1}}{dP^T_{H_0}}$ in (3.5) is independent of q. This implies that any null robust invariant test is automatically nonnull robust. Also, the optimality robustness of an invariant test follows trivially.

The crux of the problem now is to expand the R.H.S. of (3.5) in Σ locally around the null hypothesis H_0. Because of the invariance of the problem, we assume without loss of generality that Σ is of the form

$$\Sigma = \begin{bmatrix} \Lambda_1 & \Sigma_{12} \\ \Sigma_{21} & \Lambda_2 \end{bmatrix}, \quad \Lambda_1 = \mathrm{diag}(\lambda_i^*), \quad \Lambda_2 = \mathrm{diag}(\delta_i) \qquad (3.9)$$

and that the null hypothesis is specified by H_0: $\Sigma = I_p$. Local alternatives are fixed by choosing $\varepsilon > 0$ small and a suitable matrix Δ and setting $\Sigma = I_p + \varepsilon\Delta$. Writing

$$\Delta = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix},$$

with Δ_{11} and Δ_{22} as diagonal matries, we get $\Lambda_1 = I_{p_1} + \varepsilon\Delta_{11}$, $\Lambda_2 = I_{p_2} + \varepsilon\Delta_{22}$, $\Sigma_{12} = \varepsilon\Delta_{12}$ and $\Sigma_{21} = \varepsilon\Delta_{21}$. This gives

$$\Lambda_1^{-1} = I_{p_1} - \varepsilon\Delta_{11} + \varepsilon^2\Delta_{11}^2 + o(\varepsilon^2) \qquad (3.10)$$

$$\Lambda_2^{-1} = I_{p_2} - \varepsilon\Delta_{22} + \varepsilon^2\Delta_{22}^2 + o(\varepsilon^2)$$

$$(\Lambda_1 - \Sigma_{12}\Lambda_2^{-1}\Sigma_{21})^{-1} = (I_{p_1} + \varepsilon\Delta_{11} - \varepsilon^2\Delta_{12}\Lambda_2^{-1}\Delta_{21})^{-1}$$

$$= I_{p_1} - \varepsilon\Delta_{11} + \varepsilon^2(\Delta_{11}^2 - \Delta_{12}\Delta_{21}) + o(\varepsilon^2)$$

$$\Lambda_2^{-1}\Sigma_{21}(\Lambda_1 - \Sigma_{12}\Lambda_2^{-1}\Sigma_{21})^{-1}\Sigma_{12}\Lambda_2^{-1} = \varepsilon^2\Delta_{21}\Delta_{12} + o(\varepsilon^2)$$

$$\Lambda_2^{-1}\Sigma_{21}(\Lambda_1 - \Sigma_{12}\Lambda_2^{-1}\Sigma_{21})^{-1} = \varepsilon\Delta_{21} - \varepsilon^2(\Delta_{22}\Delta_{21} + \Delta_{21}\Delta_{11}) + o(\varepsilon^2)$$

Using (2.6) and (3.10), the expression for F in (3.6) is simplified as

$$F = (\mathrm{tr}S+\mathrm{tr}V)^{-1}\{\mathrm{tr}(\Sigma^{-1}-I_p)HSH' + \mathrm{tr}(\Sigma_{22}^{-1} - I_{p_2})H_2VH_2'\} \qquad (3.11)$$

$$= (\mathrm{tr}S + \mathrm{tr}V)^{-1}\{-\varepsilon[\mathrm{tr}\ \Delta_{11}H_1S_{11}H_1' + \mathrm{tr}\ \Delta_{22}H_2(S_{22}+ V)H_2'$$

$$+ 2\ \mathrm{tr}\ \Delta_{12}H_2S_{21}H_1'] + \varepsilon^2[\mathrm{tr}(\Delta_{11}^2 - \Delta_{12}\Delta_{21}H_1S_1H_1') +$$

$$+ \mathrm{tr}\ \Delta_{22}^2H_2(S_{22} + V)H_2' + \mathrm{tr}\ \Delta_{21}\Delta_{12}H_2S_{22}H_2' +$$

$$+ 2\ \mathrm{tr}(\Delta_{22}\Delta_{21} + \Delta_{21}\Delta_{11})H_1S_{12}H_2'] + o_y(\varepsilon^2)$$

where the last term $o_y(\varepsilon^2)$ in (3.11) is uniformly $o(\varepsilon^2)$ in y. We now use the following facts (Kariya (1985)):

(a) $$\int_{O(p)} \mathrm{tr}(AQBQ')\ \nu(dQ) = \frac{\mathrm{tr}\ A\ \mathrm{tr}\ B}{p}$$

(b) $$\int_{O(p)} (\mathrm{tr}\ AQBQ')^2\ \nu(dQ) = \frac{3(\mathrm{tr}\ A^2)(\mathrm{tr}\ B^2)}{p(p + 1)}$$

(c) $\quad \int_{O(p)} (\mathrm{tr}\ AQBQ')^3 (1+\theta\ \mathrm{tr} AQBQ')^{-\gamma-3} \nu(dQ) = o(\mathrm{tr}A^2)\cdot(\mathrm{tr}B^3)$

(d) $\quad \int_{O(p)} \mathrm{tr}(AQ)\ \nu(dQ) = 0$

for $0 < \theta < 1$, $\gamma > 0$ and A close to the null matrix in (c). The terms $o(\mathrm{tr}\ A^2)$ in (c) above is uniform in the elements of B.

We are now in a position to evaluate the R.H.S. of (3.6). Expanding $(1+F)^{-np/2}$ as

$$(1+F)^{-np/2} = 1-\gamma F+\gamma(\gamma+1)F^2/2-\gamma(\gamma+1)(\gamma+2)F^3(1+\theta F)^{-\gamma-3}/6 \qquad (3.12)$$

where $0 < \theta < 1$ and $\gamma = np/2$, we compute, using (3.11) and the above facts,

$$\int_{O(p_1)\times O(p_2)} F\ \nu(dH_1)\ \nu(dH_2) = \qquad (3.13)$$

$$- \varepsilon \frac{(\mathrm{tr}\ \Delta_{11})(\mathrm{tr}\ S_{11})/p_1 + (\mathrm{tr}\ \Delta_{22})(\mathrm{tr}(S_{22} + V)/p_2)}{\mathrm{tr}\ S + \mathrm{tr}\ V}$$

$$+ \varepsilon^2 \left[\frac{\{\mathrm{tr}(\Delta_{11}^2-\Delta_{12}\Delta_{21})\} \dfrac{\mathrm{tr}S_{11}}{p_1} + (\mathrm{tr}\Delta_{22}^2)(\mathrm{tr}\dfrac{(S_{22}+V)}{p_2}) + \dfrac{(\mathrm{tr}\Delta_{21}\Delta_{12})(\mathrm{tr}S_{22})}{p_2}}{\mathrm{tr}\ S + \mathrm{tr}\ V} \right]$$

$$+ o(\varepsilon^2) ;$$

$$\int_{O(p_1) \times O(p_2)} F^2 \nu(dH_1)\nu(dH_2) = (\mathrm{tr}S+\mathrm{tr}V)^{-2}\varepsilon^2\{\frac{3(\mathrm{tr}\Delta_{11}^2)\mathrm{tr}S_{11}^2}{p_1(p_1+1)} + \tag{3.14}$$

$$\frac{3(\mathrm{tr}\ \Delta_{22}^2)\ \mathrm{tr}(S_{22}+V)^2}{p_2(p_2+1)} +$$

$$4\int_{O(p_1)\times O(p_2)} (\mathrm{tr}\ \Delta_{12}H_2S_{21}H_1')^2\ \nu(dH_1)\ \nu(dH_2) +$$

$$+ 2(\mathrm{tr}\ \Delta_{11})(\mathrm{tr}\ S_{11})(\mathrm{tr}\ \Delta_{22})\ \frac{\mathrm{tr}(S_{22}+V)}{p_1p_2}\ \} + o(\varepsilon^2)\ .$$

It therefore follows that for quite general local alternatives of the type $\sum = I_p + \varepsilon\Delta$ considered above, the ratio $\frac{dP^T_{H_1}}{dP^T_{H_0}}$ is expressed as

$$\frac{dP^T_{H_1}}{dP^T_{H_0}} = |\textstyle\sum|^{-\frac{(n-1)}{2}}|\textstyle\sum_{22}|^{-\frac{1}{2}}[1+\gamma\varepsilon\ \frac{(\mathrm{tr}\Delta_{11})(\frac{\mathrm{tr}S_{11}}{p_1})+(\mathrm{tr}\Delta_{22})\frac{\mathrm{tr}(S_{22}+V)}{p_2}}{\mathrm{tr}\ S+\mathrm{tr}\ V} \tag{3.15}$$

$$+ o(\varepsilon)]$$

This means that the LBI test for a specific Δ can be obtained.

This test rejects H_0 when $\dfrac{(tr\Delta_{11})\dfrac{(trS_{11})}{p_1}+(tr\Delta_{22})\dfrac{tr(S_{22}+V)}{p_2}}{tr\ S + tr\ V}$ is large. However, because of its dependence on Δ, this result is not very useful.

On the other hand, if we consider a subclass of alternatives of the form $\sum = I_p + \varepsilon\Delta$ with $\dfrac{tr\ \Delta_{11}}{p_1} = \dfrac{tr\ \Delta_{22}}{p_2}$ then the coefficient of ε in R.H.S. of (3.15) becomes a constant and it becomes necessary to look into the coefficient of ε^2. This is readily available from the previous calculations and yields the following expression of the ratio $\dfrac{dP^T_{H_1}}{dP^T_{H_0}}$:

$$\frac{dP^T_{H_1}}{dP^T_{H_0}} = |\textstyle\sum|^{-(n-1)/2}\ |\textstyle\sum_{22}|^{-1/2}\ [1 + \gamma\varepsilon \cdot \frac{tr\ \Delta_{11}}{p_1} \tag{3.16}$$

$$+ \varepsilon^2\{(\frac{tr(\Delta^2_{11} - \Delta_{12}\Delta_{21}(tr\ S_{11})}{p_1} + \frac{(tr\ \Delta^2_{22})tr(S_{22} + V)}{p_2}$$

$$+ tr(\Delta_{21}\Delta_{12})\ \frac{tr\ S_{22}}{p_2})(trS + trV)^{-1} + (\frac{3(tr\ \Delta^2_{11})(tr\ S^2_{11})}{p_1(p_1 + 1)}$$

$$+ \frac{3(\mathrm{tr}\Delta_{22}^2)(\mathrm{tr}(S_{22}+V)^2)}{p_2(p_2+1)} + 2(\mathrm{tr}\Delta_{11})(\mathrm{tr}S_{11})(\mathrm{tr}\Delta_{22})(\mathrm{tr}(S_{22}+V))/p_1p_2$$

$$+ 4\int_{O(p_1)\times O(p_2)} (\mathrm{tr}\ \Delta_{12}H_2S_{21}H_1')^2\nu(dH_1)\nu(dH_2))(\mathrm{tr}S+\mathrm{tr}V)^{-2}\} + o(\varepsilon^2)$$

The locally best invariant test statistic against such specific local alternatives thus turns out to be the coefficient of ε^2 in the R.H.S. of (3.16). Unfortunately this again depends heavily on the fixed Δ. In the case when $\Delta_{12} = 0$, the coefficient U(say) of ε^2 simplifies to

$$U = \{\frac{(\mathrm{tr}\Delta_{11}^2)(\mathrm{tr}S_{11})}{p_1} + \frac{(\mathrm{tr}\Delta_{22}^2)(\mathrm{tr}(S_{22}+V))}{p_2}\}(\mathrm{tr}S+\mathrm{tr}V)^{-1} \qquad (3.17)$$

$$+ \{\frac{3(\mathrm{tr}\ \Delta_{11}^2)(\mathrm{tr}\ S_{11}^2)}{p_1(p_1+1)} + \frac{3(\mathrm{tr}\ \Delta_{22}^2)(\mathrm{tr}(S_{22}+V)^2)}{p_2(p_2+1)} +$$

$$+ 2(\mathrm{tr}\Delta_{11})(\mathrm{tr}\Delta_{22})(\mathrm{tr}S_{11})(\mathrm{tr}(S_{22}+V))/p_1p_2\}(\mathrm{tr}S + \mathrm{tr}V)^{-2}$$

This still depends on Δ_{11} and Δ_{22}. If we restrict Δ_{11} and Δ_{22} to satisfy

$$\frac{\mathrm{tr}\ \Delta_{11}}{p_1} = \frac{\mathrm{tr}\ \Delta_{22}}{p_2}, \qquad \frac{\mathrm{tr}\ \Delta_{11}^2}{p_1} = \frac{\mathrm{tr}\ \Delta_{22}^2}{p_2} \qquad (3.18)$$

and tr $\Delta_{11}^2 = K_1$ tr $\Delta_{22}^2 = K_2(\text{tr } \Delta_{11})(\text{tr}\Delta_{22})$

for K_1, $K_2 > 0$ known, then LBI test statistic turns out to be

$$V = \left\{\frac{3(\text{tr}S_{11}^2)}{p_1(p_1+1)} + 3\frac{\text{tr}((S_{22}+V)^2)}{K_1\ p_2(p_2+1)} + \frac{2(\text{tr}S_{11})\text{tr}(S_{22}+V)}{p_1\ p_2\ K_2}\right\} \cdot \qquad (3.19)$$

$$(\text{tr}S + \text{tr}V)^{-2}$$

The preceding analysis can be summarized as follows.

Theorem 3.1: For testing H_0: $\Sigma = \sigma^2 I_p$ vs H_1: $\Sigma = \sigma^2 I_p + \varepsilon \begin{bmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{bmatrix}$, $\varepsilon > 0$ small, in the model (2.1)-(2.2), the test which rejects H_0 for large V is LBI provided Δ_{11} and Δ_{22} satisfy (3.18).

Remark 3.2: The testing problem mentioned in Theorem 3.1 can be regarded as testing sphericity against independence.

Remark 3.3: It is interesting to observe that while the absence of V in (3.6) makes the corresponding analysis smooth and leads to an LBI test against very general local alternatives, its presence changes the problem such drastically. The test statistic V is not all that desirable because it fails to use the covariance component S_{12}.

Remark 3.4: A reasonable test for this problem would be to reject H_0 for large values of $W \equiv \frac{\text{tr } T^2}{(\text{tr } T)^2}$ where

$T = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22}+V \end{bmatrix}$. This is a generalization of the

locally optimum test statistic $\frac{\text{tr } S^2}{(\text{tr } S)^2}$ when $V = 0$ to the case when V prevails. It is possible that for some specific Δ with $\Delta_{12} \neq 0$, W may turn out to be the LBI test statistic.

Remark 3.5: It is not difficult to evaluate the integral $\int_{O(p_1) \times O(p_2)} (\text{tr } \Delta_{12}H_2S_{21}H_1')^2 \, \nu(dH_1)\nu(dH_2)$ which appears in (3.14) and (3.16). Following as in Kariya (1978), and Eaton and Kariya (1983), it can be shown that

$$\int_{O(p_1) \times O(p_2)} (\text{tr } \Delta_{12}H_2S_{21}H_1')^2 \, \nu(dH_1)\nu(dH_2) = c(\text{tr}\Delta_{12}\Delta_{21})(\text{tr}S_{21}S_{12})$$

where

$$p_1^2 c = \int_{O(p_1) \times O(p_1)} (\text{tr } H_3H_1')^2 \nu(dH_1)\nu(dH_3).$$

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